

WHEN WAITING MOVES YOU IN SCORING COMBINATORIAL GAMES

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Abstract

Combinatorial Scoring games, with the property ‘extra pass moves for a player does no harm’, are characterized. The characterization involves an order embedding of Conway’s Normal-play games. Also, we give a theorem for comparing games with scores (numbers) which extends Ettinger’s work on dicot Scoring games.

1 Introduction

The Lawyer’s offer: To settle a dispute, a court has ordered you and your opponent to play a Combinatorial game, the winner (most number of points) takes all. Minutes before the contest is to begin, your opponent’s lawyer approaches you with an offer: "You, and you alone, will be allowed a pass move to use once, at any time in the game, but you must use it at some point (unless the other player runs out of moves before you used it)." Should you accept this generous offer?

We will show when you should accept and when you should decline the offer. It all depends on whether Conway’s Normal-play games (last move wins) can be embedded in the ‘game’ in an order preserving way.

Combinatorial games have perfect information, are played by two players who move alternately, but moreover, the games finish regardless of the order of moves. When one of the players cannot move, the winner of the game is declared by some predetermined winning condition. The two players are usually called Left (female pronoun) and Right (male pronoun).

Many combinatorial games have the property that the game decomposes into independent sub-positions. A player then has the choice of playing in exactly one of the sub-positions; the whole position is the *disjunctive sum* of the sub-positions. The disjunctive sum of positions G and H is written $G + H$. Such *additive* [7] games include GO, DOMINEERING, KONANE, AMAZONS, DOTS&BOXES and also (end-positions of) CHESS but do not include HEX or any type of Maker-Maker and Maker-Breaker games. See [2], for example, for techniques to analyse these latter games.

¹Supported by the Killam Trust

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Normal-play games have the last player to move as the winner; *Misère-play* games have that player as the loser. In this paper the focus is on *Scoring games* in which the player with the greatest score wins.

Finding general results for Scoring games has proven difficult. There are only five contributors known to the authors: Milnor [8], followed by Hanner [6], considered games with no Zugzwang positions; Johnson [7] abstracted from a game played with knots; Ettinger [5, 4] considered *dicots*, that is games where either both players have a move or neither does; Stewart [12] considered a very general class of games. We give an overview of their results in Section 4.

Aviezri Fraenkel coined the terms ‘Math’ games and ‘Play’ games. The former have properties that mathematicians like. On the other hand, Play games tend to be harder to analyze, for example GO, DOTS-&-BOXES, OTHELLO, BLOKUS and KULAMI; moreover they give direction to the mathematical research. ‘Play’ Scoring games tend to have some common strategic considerations. This paper focuses on three.

- *Zugzwang* (German for “compulsion to move”) is a situation where one player is put at a disadvantage because he has to make a move when he would prefer to pass and make no move.
- *Bonus/penalty*: In many Scoring games, there are penalties or bonuses to be awarded when a game finishes.
- *Greediness principle*: Given two games G and H , Left prefers a game G for a game H , whenever each Left option of H is also a Left option in G , and each Right option of G is also a Right option of H .

Zugzwang and the Greediness principle relate to the question posed in the Lawyer’s offer. Perhaps one would believe that if all other things remain equal, giving Left an extra option is an advantage, at least no disadvantage, to Left. Surprisingly, this is not always true, indeed it is not true in [12], nor is it true in Misère-play games. If there are Zugzwangs in the ‘game’, then you would be inclined to accept, but, as we will see, this does not reveal the whole truth. See Figure 1 for an example.

Classes of Scoring games \mathcal{S} , like Normal and Misère-play games, have a defined equivalence, \equiv , (often called ‘equality’) which gives rise to equivalence classes, and where S/\equiv forms a monoid. In Normal-play games this gives an ordered abelian group. For the class of all Misère-play games the monoid has little structure. Significant results concerning Misère-play games only became possible after Plambeck and Plambeck & Siegel (see [9]) pioneered the approach of restricting the total set of games under consideration.

As shown in Stewart [12], the monoid based on the full class of Scoring games also has little structure. Following Plambeck & Siegel’s approach, we restrict the subset of Scoring games under consideration to obtain a monoid with a useful structure.

In Section 2 we formally develop the concepts needed for Scoring games, together with some basic results. In Section 5 we give some Normal-play background; see [1, 11] for more on Normal-play games.

Our main results are given in Section 3. In Theorem 2, we study when Normal-play games can be embedded in a family of Scoring games in an order preserving way. For games G and H , to check if $G \geq H$ (see Definition 6) involves comparisons using all scoring games. Theorem 4 gives a method that avoids this if one of the games is a number, and in particular answers the question: ‘who wins?’, i.e. is $G \geq 0$?

To illustrate the concepts, we refer to games based on KONANE (see also Section 2.1).

1.1 KONANE

KONANE is a traditional Hawaiian Normal-play game, played on a $m \times n$ checkerboard with white stones on the white squares and black on the black squares with some stones removed. Stones move along rows or columns but not both in the same move. A stone can jump over an adjacent opponent’s stone provided that there is an empty square on the other side, and the opponent’s stone is removed. Multiple jumps are allowed on a single move but are not mandatory. When the player to move has no more options then the game is over and, in Normal-play, the player is the loser.

SCORING-KONANE is played as KONANE, but when the player to move has no options then the game is over, and the score is *the number of stones Left has removed minus the number of stones Right has removed*. Left wins if the score is positive, loses if it is negative and ties if it is zero.

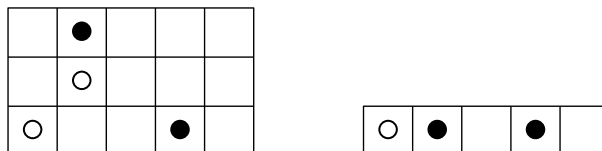


Figure 1: To the left, a SCORING-KONANE Zugzwang, for which the Lawyer’s offer should be accepted. In the right-most game, Black should reject it playing first.

In Figure 1, we show that it is not clear whether you should accept the Lawyer’s offer if you only get beforehand information that the ‘game’ to play is SCORING-KONANE, but not which particular position. In Section 2.1, we develop a variation of SCORING KONANE, with a bonus/penalty rule, for which you gladly would accept the offer, irrespective of any particular position.

2 Games terminology

Throughout, we assume best play, i.e., for example, ‘Left wins’ means that Left can force a win against all of Right’s strategies. We first give the definitions common to all variants of additive combinatorial games. Other concepts require the winning condition and these we give in separate sub-sections. We denote by $\mathbb{N}\mathbb{P}$, the set of short, Normal-play games.

The word ‘game’ has multiple meanings. Following [1, 11], when referring to a set of rules, the explicit game is in SMALL CAPITALS, otherwise, in proofs, ‘game’ and ‘position’ will be used interchangeably.

Given a game G , an *option* of G is a position that can be reached in one move; a *Left (Right) option* of G is a position that can be reached in one move by Left (Right). The set of Left, respectively Right, options of G are denoted by $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ and we write $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ to denote a typical representative of $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ respectively. A combinatorial game is defined recursively as $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$. Further, G is a *short game* if it has a form $\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$ such that $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are finite sets of short games. A game H is a *follower* of a game G if there is any sequence of moves (including the empty sequence, and not necessarily alternating) starting at G that results in the game H .

The *disjunctive sum* of the games G and H is the game $G + H$, in which a player may move in G or in H , but not both. That is, $G + H = \{G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid G^{\mathcal{R}} + H, G + H^{\mathcal{R}}\}$ where, for example, if $G^{\mathcal{L}} = \{G^{\mathcal{L}_1}, G^{\mathcal{L}_2}, \dots\}$ then $G^{\mathcal{L}} + H = \{G^{\mathcal{L}_1} + H, G^{\mathcal{L}_2} + H, \dots\}$. Clearly, the disjunctive sum operation is associative and commutative.

There is a well-known operation of ‘turning the board around’, that is, reversing the roles of both players. In Normal-play games, given a game G , this new game is the additive inverse of the old and the ‘turned’ board is denoted by $-G$ giving rise to the desirable statement $G - G = 0$, indicating that there is no advantage to either Left or Right in $G - G$. In Scoring games and also in Misère-play, the underlying structure is not necessarily a group, so for most games $G - G \neq 0$. To avoid misleading equations, we call this operation *conjugation*, and denote the *conjugate* by $\sim G$. If $G = \{G^{\mathcal{L}_1}, G^{\mathcal{L}_2}, \dots \mid G^{\mathcal{R}_1}, G^{\mathcal{R}_2}, \dots\}$ then recursively $\sim G = \{\sim G^{\mathcal{R}_1}, \sim G^{\mathcal{R}_2}, \dots \mid \sim G^{\mathcal{L}_1}, \sim G^{\mathcal{L}_2}, \dots\}$.

The notation $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$ has been identified with Normal-play games in the literature. In this paper, since we will refer to both Normal- and Scoring-play as different entities, we will use $\langle G^{\mathcal{L}} \mid G^{\mathcal{R}} \rangle$ to refer to Scoring games in order to avoid confusion.

The classes and subclasses of games that are mentioned in this paper, and in papers about Misère-play games, all have some common properties and have been given a designation.

Definition 1. Let \mathcal{U} be a set of combinatorial games. Then \mathcal{U} is a *universe* if

- (1) \mathcal{U} is closed under disjunctive sum;
- (2) \mathcal{U} is closed under taking options, that is, if $G \in \mathcal{U}$ then every $G^{\mathcal{L}} \in G^{\mathcal{L}}$ and

$G^R \in G^R$ are in \mathcal{U} ;

(3) \mathcal{U} is closed under conjugation, that is, if $G \in \mathcal{U}$ then $\sim G \in \mathcal{U}$.

2.1 Scoring games terminology

All of the previous works on Scoring games used different terminology and notation although the concepts were very similar. We unify the notation; for example, even though players sometimes have a means of keeping score during the play, the score is uniquely determined only when the player to move has no options.

Definition 2. Let G be a game with no Left options. Then we write $G^L = \emptyset^\ell$ to indicate that, if Left to move, the game is over and the score is the real number ℓ . Similarly, if $G^R = \emptyset^r$, and it is Right's move, there are no Right options and the score is r . We refer to \emptyset^s as an *atom* or, if needed for specificity, the *s-atom*. Scores will always be real numbers with the convention: Left wins if $s > 0$; Right wins if $s < 0$ and the game is a tie (drawn) if $s = 0$. Since $\langle \emptyset^s \mid \emptyset^s \rangle$ results in a score of s regardless of who moves next, we call this game s .

Games $\langle \emptyset^\ell \mid \emptyset^r \rangle$ with any of the three conditions $\ell < r$, $\ell = r$ and $\ell > r$ can occur in practice. Allowing only $\ell = r$ gives the scoring universe studied in [12].

Since addition and subtraction of real numbers does not pose a problem, we will revert to using $-s$ instead of $\sim s = \langle \emptyset^{-s} \mid \emptyset^{-s} \rangle$ for the conjugate of s .

Example 1. DISKONNECT³ is played as SCORING-KONANE, but with an additional bonus/penalty rule at the end: a piece is *insecure* if it can be captured by the opponent with a well-chosen sequence of moves (ignoring the alternating-move condition), and otherwise, the piece is *safe*. When a player to move, say Left (Black), has no more options then the game is over and Right (White) removes all of Left's insecure stones on the board. This is the penalty a player has for running out of options. The score when the game ends is *the number of stones Left has removed minus the number of stones Right has removed*.

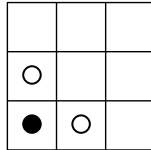


Figure 2: An endgame in KONANE, SCORING-KONANE or DISKONNECT.

³The first world championships, at TRUe games May 2014, were played on a 8×8 board with the middle 2×2 square empty. The authors placed 4th, 11th and 2nd respectively, Paul Ottaway placed 1st and Svenja Huntemann 3rd.

The game in Figure 2 is $1 = \{0 \mid \emptyset\}$ in (Normal-play) KONANE, $\langle \langle \emptyset^1 \mid \emptyset^1 \rangle \mid \emptyset^0 \rangle = \langle 1 \mid \emptyset^0 \rangle$ in SCORING-KONANE and $\langle 1 \mid \emptyset^2 \rangle$ in DISKONNECT.

In DISKONNECT, if Left moves, she jumps one stone for a score of 1 and the game is over. If it is Right to move, he has no moves and the bonus clause is invoked and since there are two white stones that could be taken, the score is 2. Observe that even though Left cannot, in play, actually take both stones, for each stone there is a legal sequence that leads to it being removed and so, each is insecure.

Scoring games can be defined in a recursive manner using the atoms.

Definition 3. The games born on Day 0 are $S_0 = \{\langle \emptyset^\ell \mid \emptyset^r \rangle : \ell, r \in \mathbb{R}\}$. For $i = 0, 1, 2, \dots$, let S_{i+1} , the games born by Day $i + 1$, be the set of games of the form $\langle \mathcal{G} \mid \mathcal{H} \rangle$, where \mathcal{G} and \mathcal{H} are non-empty finite subsets of S_i , or where either or both can be a single atom. The games in $S_{i+1} \setminus S_i$ are said to have *birthday* $i + 1$. Let $\mathbb{S} = \cup_{i \geq 0} S_i$.

We now make explicit the effect of taking a disjunctive sum of Scoring games. (See Figure 3 for an example.)

Definition 4. Given two Scoring games G and H , the disjunctive sum is given by:

$$\begin{aligned} G + H &= \langle \emptyset^{\ell_1 + \ell_2} \mid \emptyset^{r_1 + r_2} \rangle, \quad \text{if } G = \langle \emptyset^{\ell_1} \mid \emptyset^{r_1} \rangle \text{ and } H = \langle \emptyset^{\ell_2} \mid \emptyset^{r_2} \rangle; \\ &= \langle \emptyset^{\ell_1 + \ell_2} \mid G^{\mathcal{R}} + H, G + H^{\mathcal{R}} \rangle, \quad \text{if } G = \langle \emptyset^{\ell_1} \mid G^{\mathcal{R}} \rangle \text{ and } H = \langle \emptyset^{\ell_2} \mid H^{\mathcal{R}} \rangle, \\ &\quad \text{and at least one of } G^{\mathcal{R}} \text{ and } H^{\mathcal{R}} \text{ is not empty;} \\ &= \langle G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid \emptyset^{r_1 + r_2} \rangle, \quad \text{if } G = \langle G^{\mathcal{L}} \mid \emptyset^{r_1} \rangle \text{ and } H = \langle G^{\mathcal{L}} \mid \emptyset^{r_2} \rangle, \\ &\quad \text{and at least one of } G^{\mathcal{L}} \text{ and } H^{\mathcal{L}} \text{ is not empty;} \\ &= \langle G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid G^{\mathcal{R}} + H, G + H^{\mathcal{R}} \rangle, \quad \text{otherwise.} \end{aligned}$$

Note that the option $G^{\mathcal{L}} + H$ does not exist if $G^{\mathcal{L}}$ is empty (there is no addition rule for adding an empty set of options to a game). For example, consider $\langle \emptyset^1 \mid 2 \rangle + \langle 2 \mid -1 \rangle$. If Left plays, she has no move in the first component, but does have a move, so the score in the first component is not yet triggered. She must move to $\langle \emptyset^1 \mid 2 \rangle + 2$, whereupon Right moves to $2 + 2 = 4 > 0$, and Left wins. If Right plays, he should move to $\langle \emptyset^1 \mid 2 \rangle + (-1)$; now Left has no move in the sum, and the score of Left's empty set of options is triggered, giving a total score of $1 - 1 = 0$, a tie. Note also that the addition of numbers is covered by $p + s = \langle \emptyset^p \mid \emptyset^p \rangle + \langle \emptyset^s \mid \emptyset^s \rangle = \langle \emptyset^{p+s} \mid \emptyset^{p+s} \rangle$.

Game trees are a standard way to represent combinatorial games, and for Scoring games, each leaf is typically labelled with a score of game; here, if one of the players run out of moves, the node must have an atom attached to it (Figure 3). For Scoring (and Normal-play) games, we use the convention that edges down and to the right represent a Right move, those down and to the left represent a Left move.

We now turn our attention to the partial order of Scoring games. We will use Left- and Right-scores, obtained from alternating play, for comparison of Scoring games.

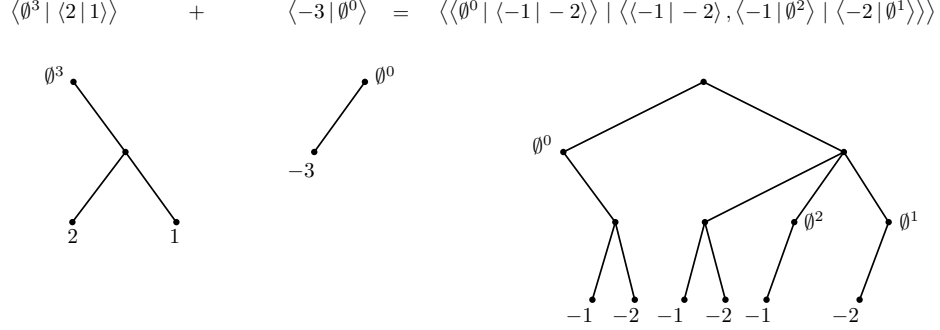


Figure 3: The disjunctive sum of two game trees.

Definition 5. The *Left-score* and the *Right-score* of a scoring game G are:

$$Ls(G) = \begin{cases} r & \text{if } G^{\mathcal{L}} = \emptyset^{\ell}, \\ \max(Rs(G^{\mathcal{L}})) & \text{otherwise;} \end{cases} \quad (1)$$

$$Rs(G) = \begin{cases} r & \text{if } G^{\mathcal{R}} = \emptyset^r, \\ \min(Ls(G^{\mathcal{R}})) & \text{otherwise.} \end{cases} \quad (2)$$

Definition 6. (Inequality in Scoring Universes)

Let $\mathcal{U} \subseteq \mathbb{S}$ be a universe of combinatorial Scoring games, and let $G, H \in \mathcal{U}$. Then $G \geq H$ if

$$Ls(G + X) \geq Ls(H + X)$$

and

$$Rs(G + X) \geq Rs(H + X),$$

for all $X \in \mathcal{U}$.

For game equivalence, we replace all inequalities in the definition, by equalities. It follows that any universe of Scoring games $\mathcal{U} \subseteq \mathbb{S}$ is a monoid, that is $0 + X = X$ for any $X \in \mathcal{U}$.

3 A natural scoring universe

Normal-play games can be regarded as Scoring games, with all scores being zero. From a mathematical perspective, it is of interest to know when they can be embedded in a scoring universe in an order preserving way.

Definition 7. Let $\mathcal{U} \subseteq \mathbb{S}$ be a universe of Scoring games with $\mathbb{R} \subset \mathcal{U}$. Define the *Normal-play mapping* $\zeta : \mathbb{Np} \hookrightarrow \mathcal{U}$ as $\zeta(G) = \widehat{G} \in \mathcal{U}$ where \widehat{G} is the game

obtained by replacing each empty set of options in the followers of $G \in \mathbb{Np}$, with the 0-atom, \emptyset^0 .

Since each atom in \widehat{G} is the 0-atom, the outcome is obviously a tie when the game is played in isolation. But the importance of the \mathbb{Np} -mapping is revealed in Definition 8, and later in Theorem 2. The mapping $f : X \rightarrow Y$ is an *order-embedding* if, for all $x_1, x_2 \in X$, $x_1 \leq x_2$ if and only if $f(x_1) \leq f(x_2)$.

Definition 8. A scoring universe \mathcal{U} is *natural* if the \mathbb{Np} -mapping ζ is an order-embedding.

Tactically, the games \widehat{n} can be regarded as *waiting-moves*, for n an integer. We will say that Left (Right) *waits* to mean that she uses one of the waiting-moves in $G + \widehat{n}$ if n is positive (negative).

For example, in the disjunctive sum $\widehat{2} + \langle -4 \mid \langle -3 \mid 5 \rangle \rangle$ Left is happy to play her waiting-move giving the option $\widehat{1} + \langle -4 \mid \langle -3 \mid 5 \rangle \rangle$; Right responds to $\widehat{1} + \langle -3 \mid 5 \rangle$; Left again waits giving $\widehat{0} + \langle -3 \mid 5 \rangle = \langle -3 \mid 5 \rangle$; Right is forced to move to 5. Further analysis shows that Left much prefers $\widehat{2} + \langle -4 \mid \langle -3 \mid 5 \rangle \rangle$ to $\widehat{1} + \langle -4 \mid \langle -3 \mid 5 \rangle \rangle$.

However, there is an even more compelling reason coming from tactical situations that appear naturally within the games we play. We invite the reader, playing Left, to contemplate which of the two games G_1 and G_2 , in Figure 4 they would prefer to add to a disjunctive sum of Scoring games.

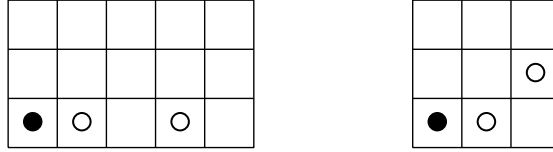


Figure 4: The DISKONNECT-games G_1 and G_2 .

From a Normal-play point of view, it is natural that more waiting-moves give a tactical advantage over fewer and so the order-embedding of Normal-play games is something to be desired if we wish to use any Normal-play intuition in Scoring play. More concretely, in G_1 , Figure 4, by ‘general principles’, Left’s jumping only one stone is clearly dominated (we invite the reader to consider the intuition of this) so

$$G_1 = \langle 2 \mid \emptyset^2 \rangle = 2 + \langle 0 \mid \emptyset^0 \rangle = 2 + \langle \langle \emptyset^0 \mid \emptyset^0 \rangle \mid \emptyset^0 \rangle = 2 + \widehat{1}$$

and

$$G_2 = \langle 1 + \langle 1 \mid \emptyset^1 \rangle \mid \emptyset^2 \rangle = 2 + \langle \langle \langle \emptyset^0 \mid \emptyset^0 \rangle \mid \emptyset^0 \rangle \mid \emptyset^0 \rangle = 2 + \widehat{2}$$

Here, G_2 , with the extra waiting-move, seems preferable over G_1 .

Games of the form $\langle \emptyset^\ell \mid G^R \rangle$ and $\langle G^L \mid \emptyset^r \rangle$ (including $\langle \emptyset^\ell \mid \emptyset^r \rangle$) will be called *atomic* in general, or *Left-atomic* and *Right-atomic*, respectively, if more precision is required.

Definition 9. Let $G \in \mathbb{S}$ be an atomic game. Then G is *stable* if $Ls(G) \leq Rs(G)$, and G is *guaranteed* if, for every atom \emptyset^s which is a follower of G^L , and every atom \emptyset^t which is a follower of G^R , $s \leq t$. In general, a game in \mathbb{S} is *stable* (*guaranteed*) if every atomic follower is stable (*guaranteed*). Let \mathbb{GS} be the class of all guaranteed Scoring games.

It is clear that \mathbb{GS} is a universe, since ‘guaranteed’ is an hereditary property; moreover, this class is also generated recursively when \mathbb{S} is generated, see Definition 3. Note that $G \in \mathbb{GS}$ implies that G is stable.

To motivate our approach for guaranteed games, we first prove an intermediate result on the stable games. In particular, the existence of ‘hot-atomic-games’, such as $\langle \emptyset^3 \mid -2 \rangle$, prevents ζ from being an order-embedding.

Theorem 1. *If \mathcal{U} is a natural universe, then each game in \mathcal{U} is stable.*

Proof. Suppose that \mathcal{U} contains a non-stable game $G = \langle \emptyset^\ell \mid G^R \rangle \in \mathcal{U}$, where $\ell > Rs(G)$. Choose k where $\ell > k > Rs(G)$ and put $X = G - k$. Consider the Left-score of $0 + X$:

$$\begin{aligned} Ls(0 + X) &= Ls(\langle \emptyset^\ell \mid G^R \rangle - k), \\ &= \ell - k > 0, \text{ since Left has no move;} \end{aligned}$$

and the Left-score of $\hat{1} + X$:

$$\begin{aligned} Ls(\hat{1} + X) &= Ls(\hat{1} + \langle \emptyset^\ell \mid G^R \rangle - k), \\ &= Rs(\langle \emptyset^\ell \mid G^R \rangle) - k, \text{ since Left only had the waiting-move,} \\ &= Rs(G) - k < 0. \end{aligned}$$

Since $Ls(0 + X) > Ls(\hat{1} + X)$, it follows from Definition 6 that 0 is not less than $\hat{1}$ and consequently \mathcal{U} is not natural. \square

Universes in which every game is stable are not necessarily natural.

Example 2. For example, in \mathbb{Np} the game 0 has many forms, and in a natural universe they all are mapped to the same game. However, in \mathbb{Np} , let $*$ = $\{0 \mid 0\}$, which gives $0 = \{*\mid*\}$. Consider the Scoring-game $G = \langle \emptyset^2 \mid \langle \langle -5 \mid 5 \rangle \mid -5 \rangle \rangle$, which is stable because $Ls(G) = 2 < Rs(G) = 5$. Now $Ls(\hat{0} + G) > 0$ and $Ls(\widehat{\{*\mid*\}} + G) < 0$, i.e. $\widehat{\{*\mid*\}} \neq \hat{0} = 0$ and the two representations of 0 are mapped to different games.

We already know that \mathbb{GS} is a stable universe, and the non-existence of hot-atomic-games ensures that it is also natural.

Theorem 2. *The universe \mathbb{GS} is natural.*

Proof. We must demonstrate that the Normal-play mapping is an order-embedding. First we show that ζ is order-preserving.

Consider $G, H \in \mathbb{Np}$ such that $G \geq H$. We want to argue that $\widehat{G} \geq \widehat{H}$ in \mathbb{GS} . The next arguments show that $Ls(\widehat{G} + X) \geq Ls(\widehat{H} + X)$, for all $X \in \mathbb{GS}$. To this purpose, we induct on the sum of the birthdays of G , H and X .

First, suppose that Left has no options in $\widehat{H} + X$. This occurs when $X = \langle \emptyset^x \mid X^R \rangle$ and $H = \{\emptyset \mid H^R\}$ (that is $\widehat{H} = \langle \emptyset^0 \mid \widehat{H}^R \rangle$). In this case, $Ls(\widehat{H} + X) = x$. Now $\widehat{G} + X = \widehat{G} + \langle \emptyset^x \mid X^R \rangle$. If Left has a move in G then, in X , which is a guaranteed game, the score will be larger than or equal to x . If Left cannot play in G then the game is over and she has a score of x . Therefore, in both cases, because the score in \widehat{G} is always 0, $Ls(\widehat{G} + X) \geq x$.

Now we assume that Left has a move in $\widehat{H} + X$. If there is a Left move, X^L , such that $Ls(\widehat{H} + X) = Rs(\widehat{H} + X^L)$, then, by induction $Rs(\widehat{H} + X^L) \leq Rs(\widehat{G} + X^L)$. By the definitions of the Left- and Right-scores, $Rs(\widehat{G} + X^L) \leq Ls(\widehat{G} + X)$, giving $Ls(\widehat{H} + X) \leq Ls(\widehat{G} + X)$. The remaining case is that there is a Left move in \widehat{H} with $Ls(\widehat{H} + X) = Rs(\widehat{H}^L + X)$. In \mathbb{Np} , $G \geq H$, i.e., $G - H \geq 0$ and so Left has a winning move in $G - H^L$. There are two possibilities, either $G^L - H^L \geq 0$ or $G - H^{LR} \geq 0$. If the first occurs, then $G^L \geq H^L$ and, by induction, $Rs(\widehat{G}^L + X) \geq Rs(\widehat{H}^L + X)$ which gives us the inequalities

$$Ls(\widehat{H} + X) = Rs(\widehat{H}^L + X) \leq Rs(\widehat{G}^L + X) \leq Ls(\widehat{G} + X).$$

If $G - H^{LR} \geq 0$ occurs, then, by induction, $Ls(\widehat{G} + X) \geq Ls(\widehat{H}^{LR} + X)$. By the definitions of Left- and Right scores, we also have $Ls(\widehat{H}^{LR} + X) \geq Rs(\widehat{H}^L + X)$ and since we are assuming that $Ls(\widehat{H} + X) = Rs(\widehat{H}^L + X)$, we can conclude that $Ls(\widehat{G} + X) \geq Ls(\widehat{H} + X)$.

The arguments to show that $Rs(\widehat{G} + X) \geq Rs(\widehat{H} + X)$ for all $X \in \mathbb{GS}$ are analogous and we omit these. Since $Ls(\widehat{G} + X) \geq Ls(\widehat{H} + X)$ and $Rs(\widehat{G} + X) \geq Rs(\widehat{H} + X)$ for all $X \in \mathbb{GS}$, then, by Definition 6, $\widehat{G} \geq \widehat{H}$.

To demonstrate that ζ is an order-embedding, it suffices to show that $G > H$ implies $\widehat{G} > \widehat{H}$ and $G \parallel H$ implies $\widehat{G} \parallel \widehat{H}$. We already know that $G > H$ implies $\widehat{G} \geq \widehat{H}$, so it suffices to show that $\widehat{G} \neq \widehat{H}$. Consider the distinguishing game $X = \sim \widehat{H} + \langle -1 \mid 1 \rangle$. We get $\widehat{H} + X = H + \sim H + \langle -1 \mid 1 \rangle$, and Left (next player) loses playing first in $H - H$. So $Ls(\widehat{H} + X) = -1$ (no player will play in the Zugzwang). But, similarly, $G > H$ implies $Ls(\widehat{G} + X) = 1$, since Left wins playing first in $G - H$. Hence $\widehat{H} \neq \widehat{G}$.

To prove that $H \parallel G$, we use the same distinguishing game $X = \sim \widehat{H} + \langle -1 \mid 1 \rangle$. We have that $-1 = Ls(H + X) < Ls(G + X) = 1$ and $1 = Rs(H + X) > Rs(G + X) = -1$, which proves the claim. \square

Some of the properties of Normal-play will hold in our Scoring universe, but there is some cost to play in a fairly general Scoring universe; there are non-invertible elements (see also Section 4). Hence comparison of games, in general, cannot be carried out as easily as in Normal-play, where $G \geq H$ is equivalent to $G - H \geq 0$ (Left win playing second in $G - H$). Here we begin by demonstrating how to compare games with scores.

Note that since $\widehat{-n} = \sim \widehat{n}$ we will revert to the less cumbersome notation $-\widehat{n}$.

Definition 10. Let G be a game in \mathbb{S} . Then $\underline{Ls}(G) = \min\{Ls(G - \widehat{n}) : n \in \mathbb{N}_0\}$ is the *Right's pass-allowed Left-score* (*pass-allowed Left-score*). The *pass-allowed Right-score* is defined analogously, $\overline{Rs}(G) = \max\{Rs(G + \widehat{n}) : n \in \mathbb{N}_0\}$.

In brief, the ‘overline’ indicates that Left can pass and the ‘underline’ that Right can pass.

Lemma 3. Let $G \in \mathbb{S}$. Then

1. $Ls(G) \geq \underline{Ls}(G)$ and $Rs(G) \leq \overline{Rs}(G)$;
2. $\overline{Rs}(G + H) \geq \overline{Rs}(G) + \overline{Rs}(H)$ and $\underline{Ls}(G + H) \leq \underline{Ls}(G) + \underline{Ls}(H)$.

Proof. The inequalities in statement 1 are obvious from the definition of the min-function, and since $G + \widehat{0} = G$.

If Left answers Right in the same component (including the possibility of a waiting-move) then this is not the full complement of strategies available to her, and this proves that $\overline{Rs}(G + H) \geq \overline{Rs}(G) + \overline{Rs}(H)$.

The inequalities for \underline{Ls} are proved similarly. \square

Definition 11. Let $\ell \in \mathbb{R}$. Then, G is *left- ℓ -protected* if $\underline{Ls}(G) \geq \ell$ and, for all G^R , there exists G^{RL} such that G^{RL} is left- ℓ -protected. Similarly, G is *right- r -protected* if $\overline{Rs}(G) \geq r$ and, for all G^L , there exists G^{LR} such that G^{LR} is right- r -protected.

The concept of ℓ -protection allows for comparisons with numbers in \mathbb{GS} .

Theorem 4. Let $G \in \mathbb{GS}$. Then $G \geq \ell$ if and only if G is left- ℓ -protected.

Proof. (\Rightarrow) Suppose that G is not left- ℓ -protected.

We will use distinguishing games of the form $X = \langle \emptyset^a \mid b - \widehat{n} \rangle$ to obtain contradictory inequalities $Rs(G + X) < 0 < Rs(\ell + X)$. The cases 1 and 2 are general considerations that don't need induction and that can be used whenever we want. We begin with the base case of the induction on the rank of G , that will be used to prove case 3.

Base case (rank 0): $G = \langle \emptyset^v \mid \emptyset^s \rangle$, with $v \leq s$.

Because G is not left- ℓ -protected, we have $Ls(G) = v < \ell$. Therefore, in order to have a distinguishing game to build the induction, we consider $X = \langle \emptyset^a \mid a + \widehat{0} \rangle$ where $-\ell < a < -v$. Then, $Rs(G + X) = v + a < 0 < a + \ell = Rs(\ell + X)$.

Case 1: $\underline{Ls}(G) = v < \ell$.

Consider $X = \langle \emptyset^a \mid a - \widehat{n} \rangle$, where $-\ell < a < -v$, and where n is large enough to obtain $\underline{Ls}(G)$. Then,

$$Rs(G + X) \leq v + a < 0$$

$$Rs(\ell + X) = \ell + a > 0$$

This is contradictory with $G \geq \ell$.

Case 2: There exists a G^R that is Left-atomic.

If $G^R = \langle \emptyset^v \mid (G^R)^R \rangle$, then consider $X = \langle \emptyset^a \mid b + \widehat{0} \rangle$ such that $a < -\ell$, $b > -\ell$ and $a < b$. Then,

$$Rs(G + X) \leq Ls(G^R + X) = v + a < 0$$

$$Rs(\ell + X) = \ell + b > 0.$$

Case 3: There exists G^R such that $\mathcal{G}^{RL} \neq \emptyset$ and all G^{RL} are not left- ℓ -protected.

Consider $G^{RL_1}, G^{RL_2}, G^{RL_3}, \dots, G^{RL_k}$; the games in \mathcal{G}^{RL} . By induction, for all $i \in \{1, \dots, k\}$, consider the distinguishing games $X_i = \langle \emptyset^{a_i} \mid b_i - \widehat{n}_i \rangle$ such that

$$Rs(G_i^{RL} + X_i) < 0 < Rs(\ell + X_i),$$

for all i .

Let $X = \langle \emptyset^{\min(a_i)} \mid \min(b_i) - \max(\widehat{n}_i) \rangle$. By convention and Theorem 2, each X_i is no better than X for Right. Hence, we get that $Rs(G_i^{RL} + X) < 0$, for all i . We also get $Rs(\ell + X) = \ell + \min(b_i) > 0$.

Therefore, $Rs(G + X) \leq Ls(G^R + X) = \max_i Rs(G_i^{RL} + X) < 0$. This contradicts $G \geq \ell$, since $Rs(\ell + X) > 0$, and thus finishes the induction step.

(\Leftarrow) Assume that G is left- ℓ -protected. We need to prove that $Rs(G + X) \geq Rs(\ell + X)$ and $Ls(G + X) \geq Ls(\ell + X) \forall X \in \mathbb{GS}$. Fix some $X \in \mathbb{GS}$ and we proceed by induction on the sum of the ranks of G and X .

Suppose there is no Right option in $G + X$, that is, $G = \langle G^L \mid \emptyset^p \rangle$ and $X = \langle X^L \mid \emptyset^q \rangle$. It follows that $Rs(G + X) = p + q$ and $Rs(\ell + X) = \ell + q$. Let $s = \underline{Ls}(G)$. Since G is left- ℓ -protected, we get $s \geq \ell$. Further, since G is guaranteed, we have that $s \leq p$, and therefore $Rs(G + X) = p + q \geq s + q \geq \ell + q = Rs(\ell + X)$.

We may now suppose that Right has a move in $G + X$. Suppose that Right's best move is in X , to say $G + X^R$. We then have the chain of relations

$$\begin{aligned} Rs(G + X) &= Ls(G + X^R), \text{ by assumption,} \\ &\geq Ls(\ell + X^R), \text{ by induction on the sum of the ranks,} \\ &= Rs(\ell + X), \text{ by definition and since numbers have empty sets of options.} \end{aligned}$$

If Right's best move is in G , to say $G^R + X$, then we have the relations

$$\begin{aligned} Rs(G + X) &= Ls(G^R + X), \text{ by assumption,} \\ &\geq Rs(G^{RL} + X), \text{ Left might have chosen a non-optimal option,} \\ &\geq Rs(\ell + X), \text{ by induction, since } G^{RL} \text{ is left-}\ell\text{-protected.} \end{aligned}$$

In all cases we have that $Rs(G + X) \geq Rs(\ell + X)$.

If Left has a move in X then let X^L be the move such that $Ls(\ell + X) = Rs(\ell + X^L)$. By the previous paragraphs, we have that $Rs(\ell + X^L) \leq Rs(G + X^L)$. Since moving to X^L may not be the best Left move in $G + X$, we know that $Rs(G + X^L) \leq Ls(G + X)$, i.e. $Ls(\ell + X) \leq Ls(G + X)$.

Now, if Left has no move in X then $X = \langle \emptyset^s | X^R \rangle$. We know that $Ls(\ell + X) = \ell + s$. In $G + X$ we restrict Left's strategy, which can only give her the same or a lower score. We denote the continuation of the two games as the G and X components. Whenever there is a Left option in the G component, Left plays the best one. If there is no option in the G component, then and only then Left plays in the X component. Any move by Right in the X component is a waiting-move in the G component so Left achieves a score of at least $\underline{Ls}(G)$; moreover, by assumption, $\underline{Ls}(G) \geq \ell$. Since X is guaranteed, the score from this component is at least s . That is, $Ls(G + X) \geq \ell + s = Ls(\ell + X)$.

For all X , we have shown that $Ls(G + X) \geq Ls(\ell + X)$ and $Rs(G + X) \geq Rs(\ell + X)$, that is, $G \geq \ell$. \square

An interesting question is: *What games, and how many, are equal to 0?* We have, for example, $G = \langle \langle 1 | 0 \rangle | \langle 0 | -1 \rangle \rangle = 0$, a game not obtained from the natural embedding. But each natural embedding of Normal-play game equal to zero maps to zero in \mathbb{GS} . Note that, for example, $\langle 0 | 0 \rangle \neq 0$, which is of course what we want since in \mathbb{Np} , $*$ = $\{0 | 0\}$ (compare this with Milnor's scoring universe in Section 4). But also $\langle \langle 1 | 0 \rangle | \emptyset^0 \rangle = 0$, a game that is not obtained from a 0 in Normal-play and neither is it a dicot.

The size of the equivalence class of 0 gives a lower bound on the sizes of the other equivalence classes since $G + H = H$ if $G = 0$. Further, note that, if $G \geq 0$, then G is Left-0-protected, and thus Right loses moving first, which is similar to the situation in Normal-play. The following corollary of Theorem 4 gives a criteria that enables us to determine when a game is equal to 0.

Corollary 5. *Let $G \in \mathbb{GS}$. Then $G = 0$ iff G is left-0- and right-0-protected, that is iff $\underline{Ls}(G) = \overline{Rs}(G) = 0$ and, for all $G^R \in G^R$, there exists $G^{RL} \geq 0$ and, for all $G^L \in G^L$, there exists $G^{LR} \leq 0$.*

4 Survey of other scoring universes

Let us begin by listing some of the game properties of the other scoring universes in the literature. In the columns, we list the authors of the three most relevant scoring universes in relation to this work; the properties are discussed, as appropriate, later in this section. The 'Yes' in Stewart's column means that there are no known 'practical methods', or that the answer is trivial as for the invertible elements. The games in Milnor's and Ettinger's universes are trivially stable, since the only atomic games are numbers. Summarizing:

Properties	Milnor	Ettinger	Stewart
Ordered Abelian Group	Yes	No	No
Equivalence Class of 0	Large	Large	Small
Invertible Elements	All	Many	Numbers
Constructive Comparisons	Yes	Yes	No
Greediness Principle	Yes	Yes	No
Game Reductions	Yes	Yes	‘Yes’
Characterization of Invertible Elements	Yes	Yes	‘Yes’
Unique Canonical Forms	Yes	No	‘Yes’
All Games Stable	‘Yes’	‘Yes’	No
Natural Embedding	No	No	No

4.1 Milnor’s non-negative incentive games

Milnor [8] and Hannor [6] considered dicot Scoring games in which there is a non-negative incentive for each player to move, and where the atoms (base of recursion) are real numbers. We denote this universe by \mathbb{PS} . Non-negative incentive translates to $Ls(G) \geq Rs(G)$, for all positions, that is *Zugzwang* situations never occur and the universe is an abelian group. As soon as $Ls(G) = Rs(G)$, the game is over and the players add up the score.

There is a similarity between these games and Normal-play games. If the scores are always integers, at the end, the players could count the score by imagining they are making ‘score’-many independent moves. Using this idea, there have been many advances in the endgame of GO (last point). Games such as AMAZONS⁴ and DOMINEERING can also be thought of as ‘territorial games’ (where the score depends on the size of captured land), but, in the literature, they have so far been analyzed under the guise of Normal-play.

Because the games in this universe have non-negative incentive and all games of \mathbb{Np} would have 0 incentive (no change in score) there is no natural embedding of \mathbb{Np} into \mathbb{PS} . In Milnor’s universe when we have the disjunctive sum of a Normal-play component with a non-negative incentive component, the outcome is determined by the non-negative incentive component. Both players want to play in the non-negative incentive component because there are no Zugzwangs. A Normal-play component alone is a tie. So, any Normal-play component is irrelevant; i.e., all embedded Normal-play components are equal to zero, and so, \mathbb{PS} is not natural.

4.2 Ettinger’s dicot Scoring games

Ettinger’s universe [5, 4] also consists of dicot games, but *Zugzwang* games like $\langle -3 \mid 3 \rangle$ are now allowed. In Ettinger’s universe, \mathbb{DS} , the atomic games are the real numbers and moreover, there are no games of the form $\langle \emptyset^r \mid \emptyset^s \rangle, r \neq s$ (although the latter game is a dicot). He noted that real numbers are not necessary, just some values taken from an ordered abelian group.

⁴see http://en.wikipedia.org/wiki/Game_of_the_Amazons

The definitions of (in)equality and of Left- and Right-scores are as in Definitions 5 and 6.

Definition 12. The disjunctive sum $G + H$ with $G, H \in \mathbb{DS}$ reduces to

$$G+H = \begin{cases} r+s & \text{if } G = r \text{ and } H = s \text{ are numbers,} \\ \langle G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid G^{\mathcal{L}} + H, G + H^{\mathcal{R}} \rangle & \text{otherwise.} \end{cases}$$

Disjunctive sum is associative and commutative, and the universe is partially ordered. The problem is that games do not necessarily have inverses. The following concept was used by Ettinger [5, p. 20-22].

Definition 13. Let $r \in \mathbb{R}$. Then, G is *left- r -safe* if $Ls(G) \geq r$ and, for all $G^{\mathcal{R}}$, there exists $G^{\mathcal{RL}}$ such that $G^{\mathcal{RL}}$ is left- r -safe. The concept of right- r -safety is defined analogously.

Two important results are:

Theorem 6 (Ettinger's Theorem). *Let $r \in \mathbb{R}$ and $G \in \mathbb{DS}$. Then, $G \geq r$ iff G is left- r -safe.*

Writing the explicit condition for 'safety', we get:

Corollary 7 (Ettinger's Corollary). *Let $G \in \mathbb{DS}$. Then $G = 0$ iff $Ls(G) = Rs(G) = 0$ and for all $G^{\mathcal{R}} \in G^{\mathcal{R}}$ there exists $G^{\mathcal{RL}} \geq 0$ and for all $G^{\mathcal{L}} \in G^{\mathcal{L}}$ there exists $G^{\mathcal{LR}} \leq 0$.*

In [5] (p. 48), it is proved that $G \in \mathbb{DS}$ is invertible iff $G+ \sim G = 0$. In fact, Ettinger proved that there are non-invertible elements and \mathbb{DS} is just a semigroup (monoid). Namely, consider $G = \langle \langle 1 \mid -1 \rangle \mid \langle 1 \mid 1 \rangle \rangle$. Then $\sim G = \langle \langle -1 \mid -1 \rangle \mid \langle 1 \mid -1 \rangle \rangle$. Now $Ls(G+ \sim G) = -2$. Therefore, using the distinguishing game 0, $Ls(G+ \sim G + 0) < Ls(0 + 0)$ and, so, by Definition 6, $G+ \sim G \neq 0$.

Lacking a group structure, how do we, constructively, know if $G \geq r$? As explained in Theorem 6, Ettinger solved the problem with the concept of r -safety.

The Greediness principle holds in \mathbb{DS} . Also, \mathbb{DS} does not have hot-atomic-games; the only games with empty sets of options are the real numbers.

Also, in [5], despite there being reductions (domination and reversibility) this does not lead to a unique canonical form for the equivalence classes.

Finally, since \mathbb{DS} is a dicot universe and \mathbb{Np} is not ($\{0 \mid \emptyset\}$ is a game in \mathbb{Np}) there is no natural order-preserving embedding from \mathbb{Np} into \mathbb{DS} . The only relation is with the subgroup of \mathbb{Np} constituted by the dicot games (formerly called *all-small*) ([1] p.185-).

Although the inclusion map from Ettinger's universe to Guaranteed Scoring is not order preserving, some nice properties still hold, using a refined setting of pass-allowed stops; see Figure 5. See also item 4 in Section 4.3.

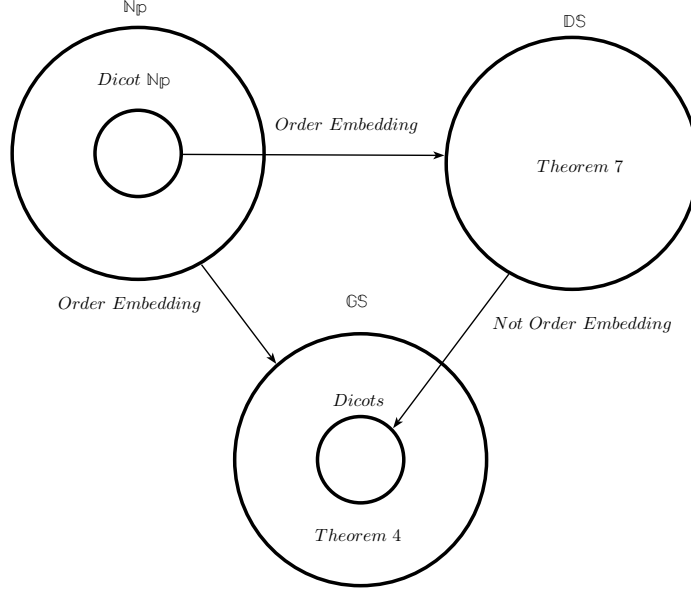


Figure 5: Inclusion maps of \mathbb{Np} , \mathbb{DS} and \mathbb{GS}

4.3 Stewart's general Scoring games

The Scoring-universe that Stewart defined [12], $\mathbb{S}' \subset \mathbb{S}$ (the only restriction is $\langle \emptyset^r \mid \emptyset^s \rangle \in \mathbb{S}'$ implies $r = s$), does allow non-stable games, in particular hot-atomic-games, such as $\langle \emptyset^9 \mid -5 \rangle$. It has some disadvantageous properties.

1) In \mathbb{S}' , we have $G = 0 \Rightarrow G \cong \langle \emptyset^0 \mid \emptyset^0 \rangle$, where ' \cong ' denotes identical game trees. (See the discussion before Corollary 5 about size of equivalence classes.) The argument is as follows. Suppose that $G = 0$ is such that $G^{\mathcal{L}} \neq \emptyset$ and consider the distinguishing game $X = \langle \emptyset^a \mid b \rangle$, where $a > 0$ and b is less than all the real numbers that occur in any follower of G . If Left starts in $G + X$ she loses; if Left starts in $0 + X$ she wins ([7], p.36). Therefore, $G \neq 0$. This situation occurs because X is a “strange” game where Left wants to have the turn but she has no moves. So, the only invertible games of \mathbb{S}' are the numbers.

2) \mathbb{S}' is non-natural. In a natural universe we have $\widehat{1} > 0$. Recall, $\zeta(1) = \langle 0 \mid \emptyset^0 \rangle = \widehat{1}$ and $\zeta(0) = \widehat{0} = 0$. Consider the distinguishing game $X = \langle \emptyset^2 \mid -3 \rangle$:

$$\begin{aligned} Ls(\langle 0 \mid \emptyset^0 \rangle + \langle \emptyset^2 \mid -3 \rangle) &= -3, \\ Ls(0 + \langle \emptyset^2 \mid -3 \rangle) &= 2, \text{ since Left has no move.} \end{aligned}$$

Thus, by Definition 6, $\hat{1} > 0$ is not true in \mathbb{S}' .

3) The Greediness principle fails in \mathbb{S}' . Consider $G = \langle 0 \mid \emptyset^0 \rangle$ and $H = \langle \emptyset^0 \mid \emptyset^0 \rangle$. There are instances when Left does not prefer G ; for example, if $X = \langle \emptyset^1 \mid -1 \rangle$, then $Ls(H + X) = 1$, $Rs(H + X) = -1$ and $Ls(G + X) = -1$, $Rs(G + X) = -1$. Thus, by Definition 6, $G \not\geq H$.

4) Clearly $\mathbb{DS} \subset \mathbb{S}$ but the inclusion map is not order-preserving (see also Figure 5 for a diagram of the results in this paper). Consider the dicot game $G = \langle \langle 1 \mid 1 \rangle \mid \langle 1 \mid 1 \rangle \rangle$. By Corollary 7, $G > 0$ in \mathbb{DS} . However, in \mathbb{S} , using the distinguishing game $X = \langle \emptyset^{\frac{1}{2}} \mid \langle \langle -2 \mid 2 \rangle \mid -3 \rangle \rangle$, of course, $Ls(0 + X) = \frac{1}{2} > 0$. But, in the game $G + X$, Left has only one legal move, that to $\langle 1 \mid 1 \rangle + X$, and so Right goes to $\langle 1 \mid 1 \rangle + \langle \langle -2 \mid 2 \rangle \mid -3 \rangle$, which gives $Ls(G' + X) = -1 < 0$. Thus, $G > 0$ in \mathbb{DS} , but $G \not\geq 0$ in \mathbb{S} .

4.4 Johnson's well-tempered, dicot Scoring games

Johnson's [7] universe consists of dicot games in which, for a given game G , the length of any play (distance to any leaf on the game tree) has the same parity. The games are called *even-tempered* if all the lengths are even and called *odd-tempered* otherwise. A game, G is *inversive* if $Ls(G + X) \geq Rs(G + X)$ for every even-tempered game X . Although the whole set of games is not well-behaved, for example, canonical forms do not exist, each inversive game has a canonical form and an additive inverse which is equal to its conjugate; in fact they form an abelian group. Moreover, $G \geq H$ if G and H have the same 'temper' and $Rs(G - H) \geq 0$.

5 Normal-play games

The definitions for Normal-play are standard, along with other material, in any of [3, 1, 11].

Under Normal-play, there are four outcome classes:

Class	Name	Definition
\mathcal{N}	incomparable	The \mathcal{N} ext player wins
\mathcal{P}	zero	The \mathcal{P} revious player wins (more precisely \mathcal{N} ext player loses)
\mathcal{L}	positive	\mathcal{L} eft wins regardless of who plays first
\mathcal{R}	negative	\mathcal{R} ight wins regardless of who plays first

We write $\circ(G)$ to designate the outcome of G . The fundamental definitions of Normal-play structure are based in these outcomes.

Definition 14. (Equivalence) $G = H$ if $\circ(G + X) = \circ(H + X)$ for all games X .

The convention is that positive is good for Left and negative for Right and the outcomes are ordered: \mathcal{L} is greater than both \mathcal{N} and \mathcal{P} , which in turn are both

greater than \mathcal{R} , and, finally, \mathcal{N} and \mathcal{P} are incomparable. In Normal-play games, there is a way to check for the equivalence of games G and H , which does not require considering any third game:

$$G = H \text{ iff } G - H \text{ is a second player win.}$$

Definition 15. (Order) $G \geq H$ if $\forall X, o(G + X) \geq o(H + X)$.

Definition 16. (Number) A game G is a number if all G^L and G^R are numbers and for each pair of options, $G^L < G^R$.

Note that $0 = \{\emptyset \mid \emptyset\}$ is a number since there are no options to compare. If the number is positive, then this represents the number of moves advantage Left has over Right. An important concept is the ‘stop’—the best number that either player can achieve when going first.

Definition 17. (Stops) The *Left-stop* and the *Right-stop* of a game G are:

$$LS(G) = \begin{cases} G & \text{if } G \text{ is a number,} \\ \max(RS(G^L)) & \text{if } G \text{ is not a number;} \end{cases} \quad (3)$$

$$RS(G) = \begin{cases} G & \text{if } G \text{ is a number,} \\ \min(LS(G^R)) & \text{if } G \text{ is not a number.} \end{cases} \quad (4)$$

There are two possible relations between the stops. One: $LS(G) > RS(G)$ and G is referred to as a *hot* game. The term was chosen to give the idea that the players really want to play first in G . Two: $LS(G) = RS(G)$ and G is either a number or a *tepid* game such as a position in NIM and CLOBBER [1]. In the presence of a hot game, moves in a tepid game are not urgent, and moves in numbers are never urgent. The situation $LS(G) < RS(G)$ does not occur, since then G would be a Zugzwang game, but in Normal-play these games are numbers and so the relationship reverts to $LS(G) = RS(G)$.

The Lawyer’s offer should obviously have been accepted if the question would have concerned Normal-play, because here the worst thing imaginable is to run out of options.

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